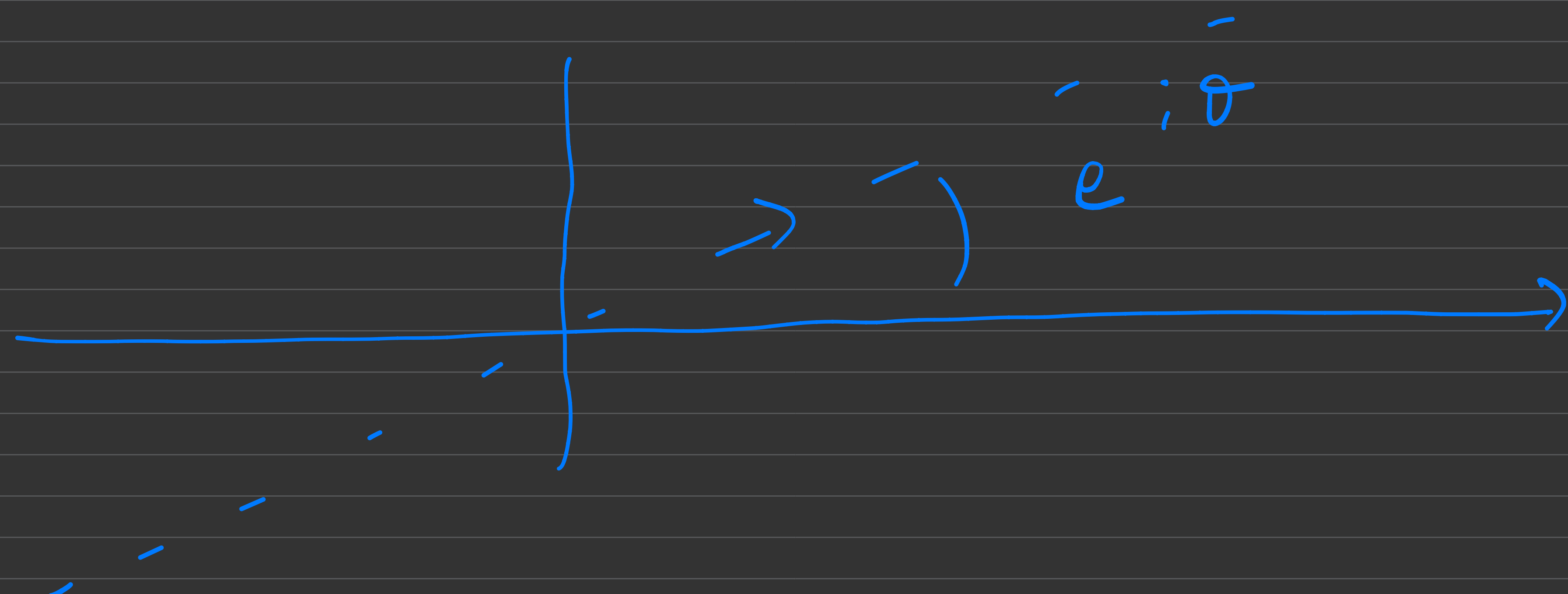
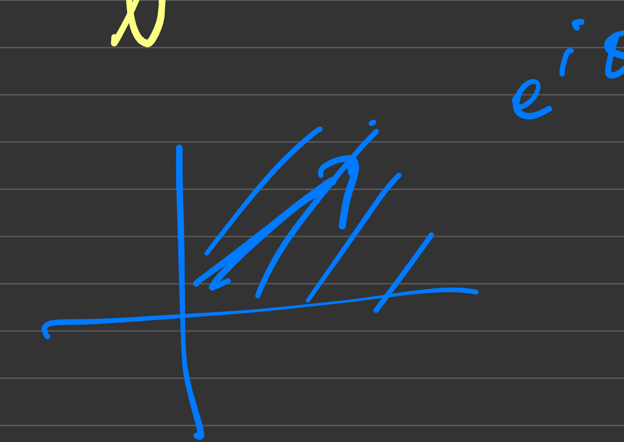
Euclidean $\Longleftrightarrow$ Minkowsky

$$Z[J,\theta] = \int \prod_i dq_i \, e^{-\sum_i \frac{1}{2}\frac{(\Delta q_i)^2}{\delta e^{i\theta}} + V(q_i)\delta e^{i\theta} - J_i q_i \delta}$$

$$\frac{t_i - t_f}{N} \equiv \delta$$





$$= \int \mathcal{D}q \, e^{-\int \left(\frac{e^{-i\theta}}{2}\dot{q}^2 + e^{i\theta}V(q) - Jq\right)d\tau}$$

$$0 < \theta < \frac{\pi}{2}$$

$\theta = 0 \Rightarrow$ Euclidean path integral

$\theta \to \frac{\pi}{2}^{-} \equiv$ Minkowsky (with ε - prescription)

$\cdot \; \theta = \frac{\pi}{2} - \varepsilon \qquad e^{i\theta} = i(1 - i\varepsilon) \qquad e^{-i\theta} = -i(1 + i\varepsilon)$

$$\to \int \mathcal{D}q \; e^{i\int \frac{\dot{q}^2}{2} - V(q) + \frac{i\varepsilon}{2}\left(\dot{q}^2 + V(q)\right) - \int J q}$$

Ex $V(q) = \frac{m^2 q^2}{2} \Rightarrow \frac{1}{\omega^2 - m^2} \Rightarrow \frac{1}{\omega^2(1+i\varepsilon) - m^2(1-i\varepsilon)}$

$$\simeq \frac{(1 - i\varepsilon)}{\omega^2 - m^2 + 2i\varepsilon m^2}$$

$+i\varepsilon$

Analytic continuation

$t_i \ll 0 < \tau \ll t_f$

Ex 2 pt function

$$\langle q(\tau)\, q(0) \rangle_\theta = \lim_{\substack{\delta \to 0 \\ n\delta = \tau}} \int \prod_i Dq_i \; \underbrace{q_n \, q_0}\; e^{-\sum_i \frac{(\Delta q_i)^2}{\delta e^{i\theta}} + V(q_i)\,\delta e^{i\theta}}$$

$$\frac{t_f - t_i}{N} = \delta$$

$$\lim_{\substack{\delta \to 0 \\ n\delta = \tau}} f(n, \delta e^{i\theta}) = g(|n|\,\delta\, e^{i\theta}) = g(|\Delta\tau|\, e^{i\theta})$$

$$\langle q(\tau)\, q(0) \rangle_E \equiv \langle q(\tau)\, q(0) \rangle_{\theta=0} = f(|\Delta\tau|) = G(\Delta\tau)$$

$$\langle q(\tau)\, q(0) \rangle_M = \langle q(\tau)\, q(0) \rangle_{\theta \to \pi/2} = g(|\Delta\tau|\, i)$$

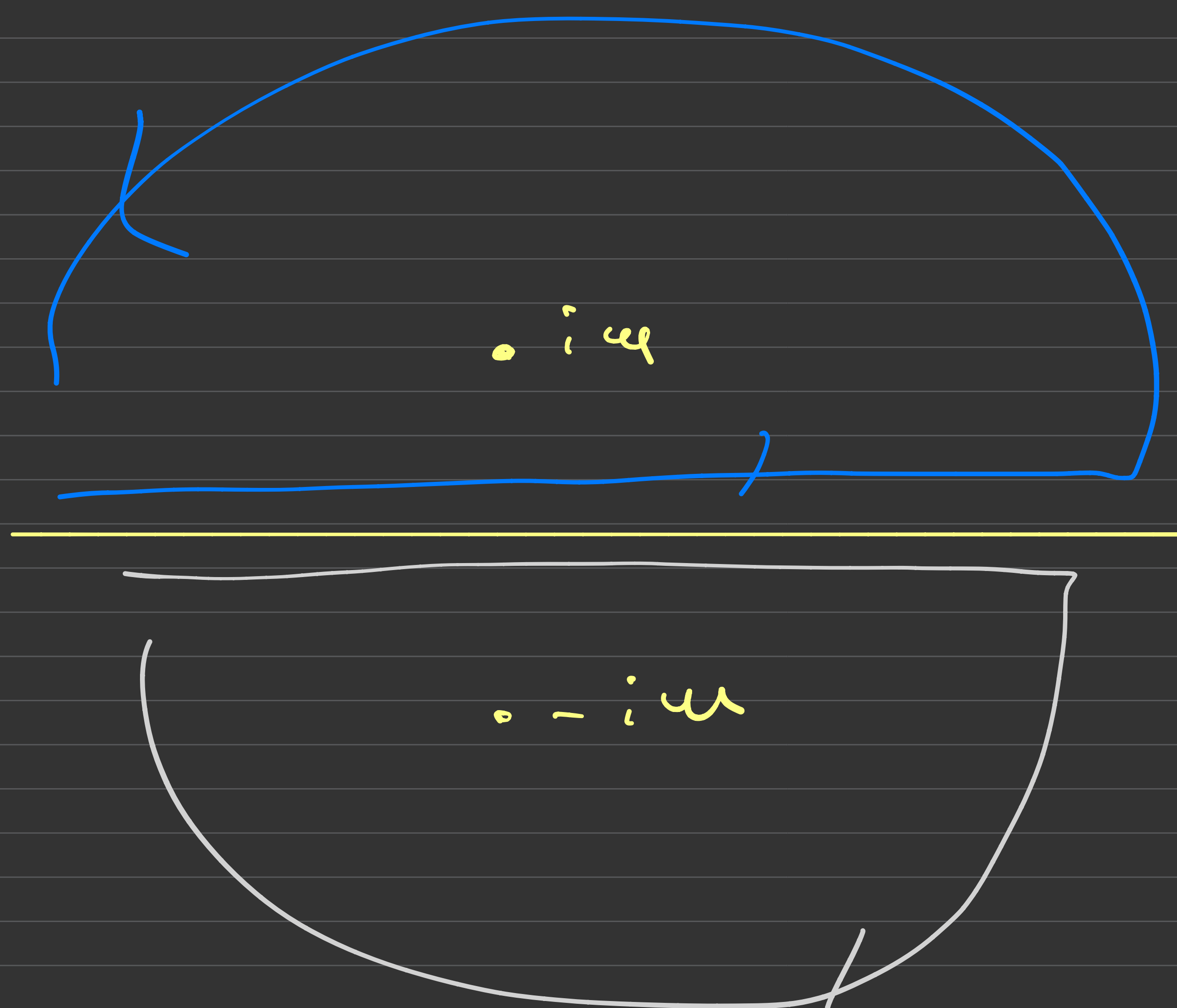
$\Delta\tau > 0$ $$f(\Delta\tau) \longrightarrow f(\Delta\tau \cdot i)$$

$\Delta\tau < 0$ $$f(-\Delta\tau) \longrightarrow f(-\Delta\tau \cdot i)$$

$$\Delta\tau \to \Delta\tau \cdot i \equiv \underline{\text{Wick}}\ \underline{\text{rotation}}$$

Ex Harmonic oscillator $\mathcal{L}_E = \frac{\dot{q}^2}{2} + \frac{m^2}{2} q^2$

$$G(\tau) = \frac{1}{2m} e^{-m|\tau|} = \int \frac{d\omega}{2\pi} \frac{e^{-i\omega\tau}}{\omega^2 + m^2}$$



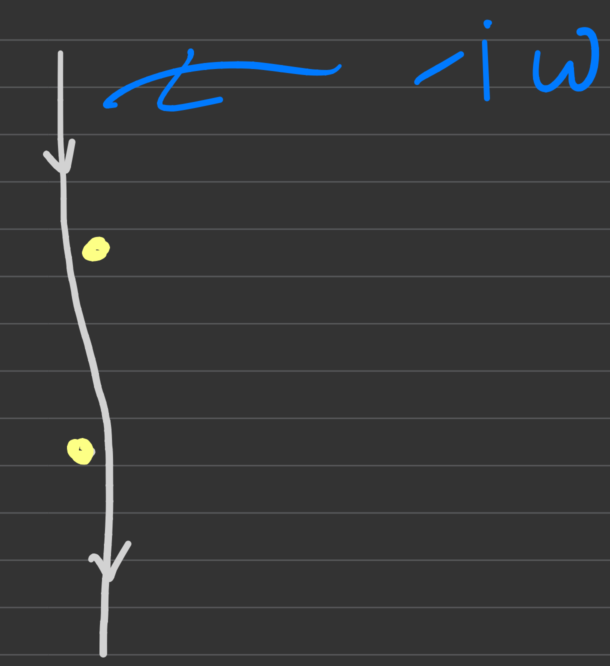
$$\left.\begin{array}{l} \tau < 0 \implies \sim e^{m\tau} \\ \tau > 0 \implies \sim e^{-m\tau} \end{array}\right\} e^{-m|\tau|}$$

$$G_\theta(\tau) = \frac{1}{2m} e^{-m|\tau| e^{i\theta}} \equiv \text{same poles with no contribution at } \infty$$

$$\Rightarrow \int d\omega \, \frac{e^{-i\omega\tau}}{\omega^2 + m^2}$$

$$\tau \to e^{i\theta}\tau$$
$$\omega \to e^{-i\theta}\omega$$

$$\Longrightarrow$$



$$= \int \frac{-i\,d\omega}{2\pi} \frac{e^{-i\omega\tau}}{-\omega^2 + m^2 - i\varepsilon} = \int \frac{d\omega}{2\pi} \frac{i\,e^{-i\omega\tau}}{\omega^2 - m^2 + i\varepsilon}$$

S-matrix

1) Compute euclidean correlators by path integral

2) Wick rotate to Minkowsky and use LSZ

- Ground state projection

$$\langle\psi_f| e^{-H(\tau_f-\tau_n)} \mathcal{O}_n(x_n) e^{-H(\tau_n-\tau_{n-1})} \cdots \mathcal{O}_1(x_1) e^{-H(\tau_1-\tau_i)} |\psi_i\rangle$$

$$\Downarrow \quad \tau_f-\tau_i \to \infty$$

$$\underbrace{\langle\psi_f|0\rangle\langle 0|\psi_i\rangle e^{-E_0(\tau_f-\tau_i)}}_{=\langle\psi_f| e^{-H(\tau_f-\tau_i)}|\psi_i\rangle}\left(\langle 0| T\left(\mathcal{O}_n(x_n)\cdots\mathcal{O}_1(x_1)\right)|0\rangle + \mathcal{O}\left(e^{-(E_1-E_0)(\tau_f-\tau_i)}\right)\right)$$

$$\Rightarrow \langle 0|T\left(\mathcal{O}_n(x_n)\cdots\mathcal{O}_1(x_1)\right)|0\rangle = \lim_{\substack{\tau_i\to-\infty \\ \tau_f\to+\infty}} \left[\frac{\int_{\psi_i\psi_f} \mathcal{D}\varphi\, \mathcal{O}_n(x_n)\cdots\mathcal{O}_1(x_1)\, e^{-S}}{\underbrace{\int_{\psi_i\psi_f}\mathcal{D}\varphi\, e^{-S}}}\right]$$

$$D\varphi \Big|_{\text{saddles}} = \frac{D\varphi}{\int\limits_{q_i, q_f} D\varphi \, e^{-S}}$$

$$\langle 0| T(\hat{O}_1 \ldots \hat{O}_n) |0\rangle = \lim_{\substack{\tau_i \to -\infty \\ \tau_f \to +\infty}} \int D\varphi \, O_1 \ldots O_n \, e^{-S}$$

Functional Methods

- partition function: $$Z[J] = \int \mathcal{D}\phi \; e^{-S + J\phi}$$

$$J\phi \equiv \int J_a(x)\,\phi_a(x)\,d^4x$$

$$\boxed{\frac{1}{Z[0]}\frac{\delta}{\delta J_1(x_1)}\cdots\frac{\delta}{\delta J_n(x_n)} Z[J]\bigg|_{J=0} = \frac{\int \mathcal{D}\phi\; \phi_n(x_n)\cdots\phi_1(x_1)\, e^{-S}}{\int \mathcal{D}\phi \; e^{-S}} = \langle 0| T\big(\phi_n(x_n)\cdots\phi_1(x_1)\big)|0\rangle}$$

- Connected correlators

- <u>definition</u>

$$\langle \phi_{a_1}(x_1) \cdots \phi_{a_n}(x_n) \rangle \equiv \sum_{\text{All partitions}} \langle \quad \rangle_c \langle \quad \rangle_c \cdots$$

- $\langle \phi_{a_1}(x_1) \rangle = \langle \phi_{a_1}(x_1) \rangle_c$

- $\langle \phi_{a_1}(x_1) \phi_{a_2}(x_2) \rangle = \langle \phi_{a_1}(x_1) \phi_{a_2}(x_2) \rangle_c + \langle \phi_{a_1}(x_1) \rangle_c \langle \phi_{a_2}(x_2) \rangle_c$

$$\langle \phi_a \phi_b \rangle_c = \langle \phi_a \phi_b \rangle - \langle \phi_a \rangle \langle \phi_b \rangle$$

More compactly $\phi_a(x) : (x, a) \longrightarrow a$

$$\sum_a \int d^4x \longrightarrow \sum_a$$

$$\langle \phi_a \phi_b \phi_c \rangle = \langle \phi_a \phi_b \phi_c \rangle_c + \langle \phi_a \rangle \langle \phi_b \phi_c \rangle_c + \langle \phi_b \rangle \langle \phi_a \phi_c \rangle_c$$

$$+ \langle \phi_c \rangle \langle \phi_a \phi_b \rangle_c$$

$$+ \langle \phi_a \rangle \langle \phi_b \rangle \langle \phi_c \rangle$$

$$\cancel{Z[0]}\, \langle \phi_{a_1} \cdots \phi_{a_n} \rangle = \frac{\delta}{\delta J_{a_1}} \cdots \frac{\delta}{\delta J_{a_n}} Z[J] \Big|_{J=0}$$

$$\langle \phi_{a_1} \cdots \phi_{a_n} \rangle_c = \frac{\delta}{\delta J_{a_1}} \cdots \frac{\delta}{\delta J_{a_n}} \ln Z[J] \Big|_{J=0}$$

$$W \equiv \ln Z$$

- $Z[J] \equiv e^{W[J]}$ ⟶ generator of <u>connected</u> <u>correlators</u>

- $Z[0] = 1 \implies W[0] = 0$

- $$\langle \phi_1 \cdots \phi_n \rangle = \frac{\delta}{\delta J_1} \cdots \frac{\delta}{\delta J_n} Z \Big|_{J=0} = \frac{\delta}{\delta J_1} \cdots \frac{\delta}{\delta J_n} e^{W[J]} \Big|_{J=0}$$

$$= \left(\sum_{\substack{\text{All} \\ \text{partitions}}} \frac{\delta}{\delta J} W \right)$$

<u>Ex</u>

$$\langle \phi_1 \phi_2 \rangle = \frac{\delta}{\delta \phi_1} \frac{\delta}{\delta \phi_2} W + \frac{\delta W}{\delta \phi_1} \frac{\delta W}{\delta \phi_2}$$

- Effective action (sketch)

$$\int \mathcal{D}\phi \; e^{-S[\phi] + J\phi}$$

$$-\frac{\delta S}{\delta \phi} + J = 0 \quad \Rightarrow \quad \phi[J] = \phi_c$$

$$J = J[\phi_c]$$

$$\sim e^{-S[\phi[J]] + J\phi[J]} \left(1 + \cdots \right) = Z[J] = e^{W[J]}$$

$\hbar \to 0$

$$-S[\phi[J]] + J\phi[J] = \underline{W[J]}$$

$$-S[\phi_c] + J[\phi_c]\phi_c = W[J[\phi_c]]$$

$$\underline{S[\phi_c]} = \underline{-W[J[\phi_c]]} + J[\phi_c]\phi_c$$

Effective Action

$$W[J] \xrightarrow{\text{Legendre}} \qquad \phi_a = \frac{\delta W}{\delta J_a} \qquad \phi_a = \phi_a[J]$$

def

$$\Gamma[\phi] = -W[J[\phi]] + J[\phi]\,\phi$$

$J_b \phi_b$

quantum effective action

- $$\frac{\delta \Gamma}{\delta \phi_b} = -\frac{\delta W}{\delta J_a}\frac{\delta J_a}{\delta \phi_b} + J_b + \phi_a \frac{\delta J_a}{\delta \phi_b}$$

$$= -\cancel{\frac{\delta W}{\delta J_a}\frac{\delta J_a}{\delta \phi_b}} + J_b + \cancel{\frac{\partial W}{\partial J_a}\frac{\delta J_a}{\delta \phi_b}} = J_b$$

Cluster property

- In euclidean QFT : $\langle O_1(x_1)\cdots O_n(x_n)\, O_{n+1}(x_{n+1}+a)\cdots O_{n+m}(x_{n+m}+a)\rangle$

$$\xrightarrow{a\to\infty} \langle O_1(x_1)\cdots O_n(x_n)\rangle \langle O_{n+1}(x_{n+1})\cdots O_{n+m}(x_{n+m})\rangle$$

- by induction $\Longrightarrow$

$$\langle O_1(x_1)\cdots O_n(x_n)\, O_{n+1}(x_{n+1}+a)\cdots O_{n+m}(x_{n+m}+a)\rangle_c \xrightarrow{a\to\infty} 0$$

2-pt

$$\langle O_1(x_1)\, O_2(x_2)\rangle = \langle O_1(x_1)\, O_2(x_2)\rangle_c + \langle O_1\rangle\langle O_2\rangle$$

cluster $\Rightarrow$ $\lim_{x_2\to\infty} \langle O_1(x_1) O_2(x_2)\rangle = \langle O_1\rangle\langle O_2\rangle$

3 pt

$$\langle O_1(x_1)\, O_2(x_2)\, O_3(x_3)\rangle = \langle O_1(x_1)\, O_2(x_2)\, O_3(x_3)\rangle_c +$$

$$+ \langle O_1\rangle\langle O_2(x_2)\, O_3(x_3)\rangle_c + \langle O_2\rangle\langle O_1(x_1)\, O_3(x_3)\rangle_c + \langle O_3\rangle\langle O_1(x_1)\, O_2(x_2)\rangle_c$$

$$+ \langle O_1\rangle\langle O_2\rangle\langle O_3\rangle$$

$$\langle O_1\rangle\left(\langle O_2\, O_3\rangle_c + \langle O_2\rangle\langle O_3\rangle\right) = \langle O_1\rangle\langle O_2\, O_3\rangle$$

- $\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle_c \longrightarrow 0$ when any $|x_i - x_j| \to \infty$

$\Longrightarrow$ Fourier transf "well behaved at" $p_i \to 0$

$$= \int e^{ip_1 x_1 - \cdots ip_n x_n} \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle \, d^4x_1 \cdots d^4x_n$$

$$= (2\pi)^4 \delta^4(p_1 + \cdots p_n) \, G_c(p_2, \ldots p_n)$$

$$G_c(0, x_2, \ldots, x_n) = \int e^{-ip_2 x_2 + \cdots - ip_n x_n} \underbrace{G_c(p_2, \ldots p_n)} \, d^4p_2 \cdots d^4p_n$$

$$\langle \phi_1(x_1)\cdots\phi_n(x_n)\rangle = \langle \phi_1(x_1)\cdots\phi_n(x_n)\rangle_c +$$

$$+\langle\phi_1(x_1)\,\phi_2(x_2)\rangle_c\,\langle\phi_3(x_3)\cdots\phi_n(x_n)\rangle_c + \cdots$$

$$\delta^4(p_1+p_2) \qquad \delta^{(4)}(p_3+\cdots p_n)$$

$$\equiv \underbrace{\delta^{(4)}(p_1+\cdots p_n)}\,\underbrace{\delta^{(4)}(p_1+p_2)}$$

ex 2pt function $m^2 = 0$

$$G_c(p^2) = \frac{1}{p^2}$$

$$G_c(0, x) = \int e^{-ipx} \frac{1}{p^2} d^4p \sim \frac{1}{x^2}$$

if $G_c(p) = \delta^4(p)$

$$G_c(0, x) \longrightarrow \text{const}$$

* Effective action (sketch)

$$\int \mathcal{D}\phi \, e^{-S[\phi]+J\phi} = Z[J]$$

$$-\frac{\delta S}{\delta \phi} + J = 0 \Rightarrow \phi[J] = \phi_c$$

$$J = J[\phi_c]$$

$$\sim e^{-S[\phi[J]]+J\phi[J]} \left(1 + \ldots\right) = Z[J] = e^{W[J]}$$

semiclassical approx $\quad W[J] = -S[\phi[J]] + J\cdot\phi[J] \ldots$

$\Downarrow$

$W[J] \equiv$ Legendre transf of $S[\phi]$

$W[J] \Rightarrow$ define $\Gamma[\phi] \equiv$ Legendre transf of $W[J]$

Effective Action

$$W[J] \xrightarrow{\text{Legendre}} \qquad \tilde{\phi}_a = \frac{\delta W}{\delta J_a} \qquad \tilde{\phi}_a = \tilde{\phi}_a[J]$$

def

$$\Gamma[\tilde{\phi}] = -W[J[\tilde{\phi}]] + \underbrace{J[\tilde{\phi}]\,\tilde{\phi}}_{J_b \tilde{\phi}_b}$$

quantum effective action

- $$\frac{\delta \Gamma}{\delta \tilde{\phi}_b} = -\frac{\delta W}{\delta J_a}\frac{\delta J_a}{\delta \tilde{\phi}_b} + J_b + \tilde{\phi}_a \frac{\delta J_a}{\delta \tilde{\phi}_b}$$

$$= -\cancel{\frac{\delta W}{\delta J_a}\frac{\delta J_a}{\delta \tilde{\phi}_b}} + J_b + \cancel{\frac{\partial W}{\partial J_a}\frac{\delta J_a}{\delta \tilde{\phi}_b}} = J_b[\tilde{\phi}]$$

- $$\tilde{\phi}_a = \frac{\delta W[J]}{\delta J_a} = \langle \phi_a \rangle_J \xrightarrow{J=0} \langle 0 | \phi_a | 0 \rangle$$

- $$J_a = \frac{\delta \Gamma[\tilde{\phi}]}{\delta \tilde{\phi}_a} \xrightarrow{J=0} \left. \frac{\delta \Gamma}{\delta \tilde{\phi}_a} \right|_{\tilde{\phi}_a = \langle \phi_a \rangle} = 0$$

⓪ Γ - generator of proper vertices

2-point function

$$\left.\frac{\delta^2\Gamma}{\delta\phi_a\,\delta\phi_b}\right|_{\phi=\langle\phi\rangle} = \frac{\delta}{\delta\phi_a}\frac{\delta\Gamma}{\delta\phi_b} = \frac{\delta J_b}{\delta\phi_a} = \left.\left(W^{(2)-1}\right)_{ba}\right|_{J=0}$$

$$\delta_{bc} = \frac{\delta J_b}{\delta\phi_a}\frac{\delta\phi_a}{\delta J_c} = \frac{\delta J_b}{\delta\phi_a}\cdot\frac{\delta^2 W}{\delta J_a\,\delta J_c}$$

$$= \frac{\delta J_b}{\delta\phi_a}\cdot W^{(2)}_{ac}$$

$$\left(W^{(2)-1}\right)_{ba} = \frac{\delta J_b}{\delta\phi_a}$$

$$\boxed{\Gamma^{(2)}_{ab} = \left(W^{(2)}\right)^{-1}_{ab}}$$

$$\left(W^{2}\right)^{-1}_{ab} \neq \frac{1}{W^{(2)}_{ab}}$$

$$\Gamma^{(2)}_{ab} \, W^{(2)}_{bc} = \delta_{ac}$$

Ex: free field $a, b, c, \ldots \longrightarrow p \equiv$ momenta

$$W_{bc} \longrightarrow \underbrace{\delta^4(p_1 - p_2)\, G(p_1)} \equiv G(p_1, p_2)$$

$$\Gamma^{(2)}_{ab} \longrightarrow \Gamma(p_1, p_2)$$

$$\int \Gamma(p_1, p_2)\, G(p_2, p_3)\, d^4p_2 = \delta^4(p_1 - p_3)$$

$$\Rightarrow \Gamma(p_1, p_2) = \delta^4(p_1 - p_2)\, G^{-1}(p_1)$$

$$\int G^{-1}(p_1)\,\delta^4(p_1-p_2)\,\delta^4(p_2-p_3)\,G(p_3)\,d^4p_2$$

$$\int G^{-1}(p_1)\,\partial^4(p_1-p_3)\,G(p_3)=\delta(p_1-p_3)$$

$$\left.\begin{aligned} \frac{\delta J_b}{\delta \phi_a} \cdot \frac{\delta \phi_a}{\delta J_c} &= \delta_{bc} \\ \frac{\delta J_b}{\delta \phi_a} \cdot W^{(2)}_{ac} &= \delta_{bc} \end{aligned}\right\} \Rightarrow \frac{\delta J_b}{\delta \phi_a} = \left(W^{(2)-1}\right)_{ba}$$

- 3 pt function

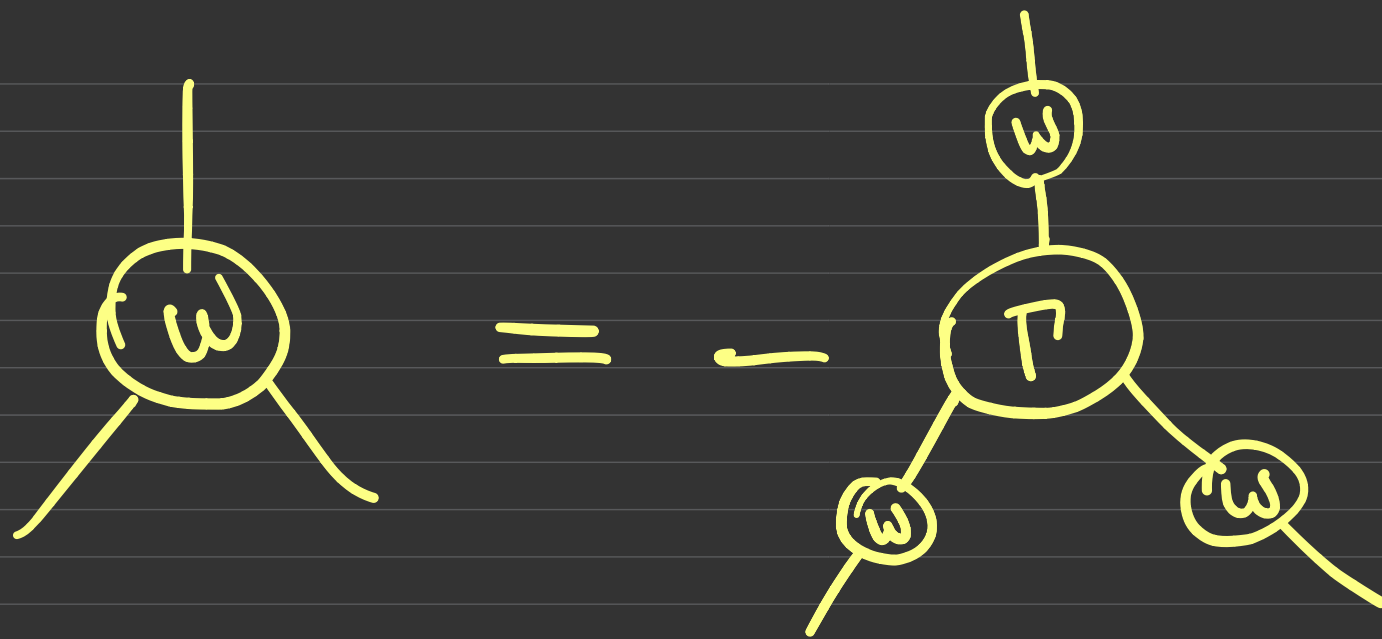
$$\frac{\delta^3 \Gamma}{\delta\phi_a \delta\phi_b \delta\phi_c} = \overbrace{\frac{\delta}{\delta\phi_c}\left(\frac{\delta^2 W}{\delta J \delta J}\right)^{-1}_{ab}}^{\frac{\delta}{\delta\phi_c}\frac{\delta^2\Gamma}{\delta\phi_a\delta\phi_b}} = \frac{\delta}{\delta\phi_c}\frac{\delta^2 W}{\delta J_{a'}\delta J_{b'}}$$

$$= -\left[\left(W^{(2)}\right)^{-1}\frac{\delta W^{(2)}}{\delta\phi_c}\left(W^{(2)}\right)^{-1}\right]_{ab}$$

$$\Gamma^{(3)}_{abc} = -\left(W^{(2)}\right)^{-1}_{aa'}\left(W^{(2)}\right)^{-1}_{bb'}\left(W^{(2)}\right)^{-1}_{cc'} W^{(3)}_{a'b'c'}$$

$$W^{(3)}_{abc} \equiv \frac{\delta^3 W}{\delta J_a \delta J_b \delta J_c}$$

$$\boxed{W^{(3)}_{abc} = -W^{(2)}_{aa'} W^{(2)}_{bb'} W^{(2)}_{cc'} \Gamma^{(3)}_{a'b'c'}}$$

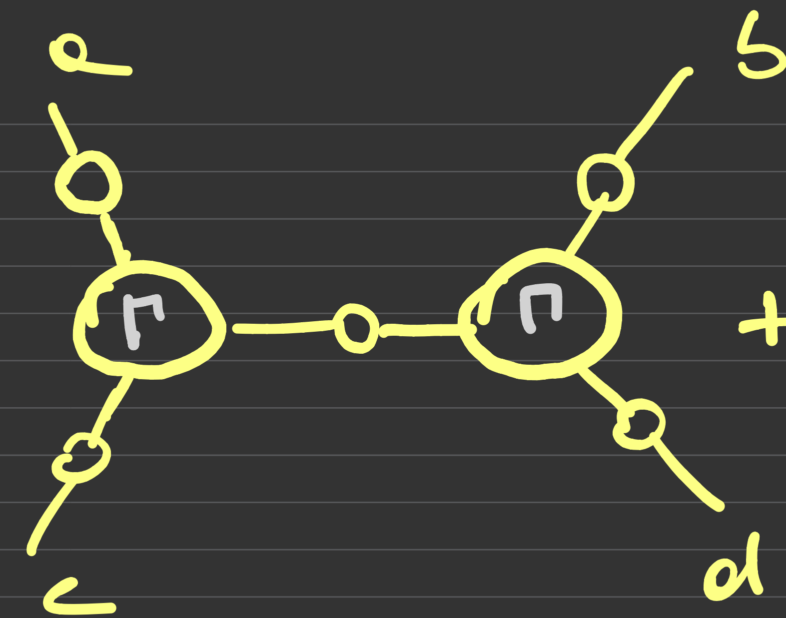


4-pt $\Longrightarrow$

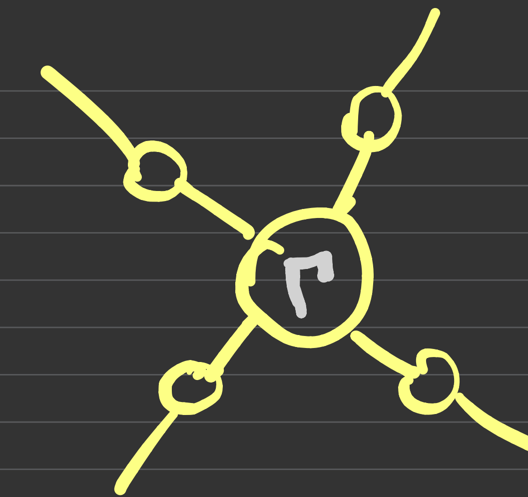
$$W^{(4)}_{abcd} \equiv \frac{\delta}{\delta J_d} W^{3}_{abc} = -\Big[W^{(3)}_{daa'} W^{(2)}_{bb'} W^{(2)}_{cc'} \Gamma^{(3)}_{a'b'c'} + \text{perms}$$

$$+ W^{(2)}_{aa'} W^{(2)}_{bb'} W^{(2)}_{cc'} W^{(2)}_{dd'} \Gamma^{(4)}_{a'b'c'd'} \Big]$$

$=$



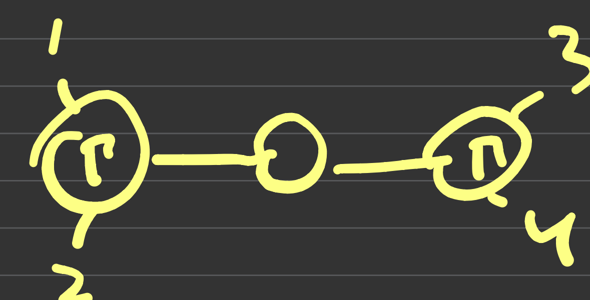
$+$ perms $-$



$2 \to 2$

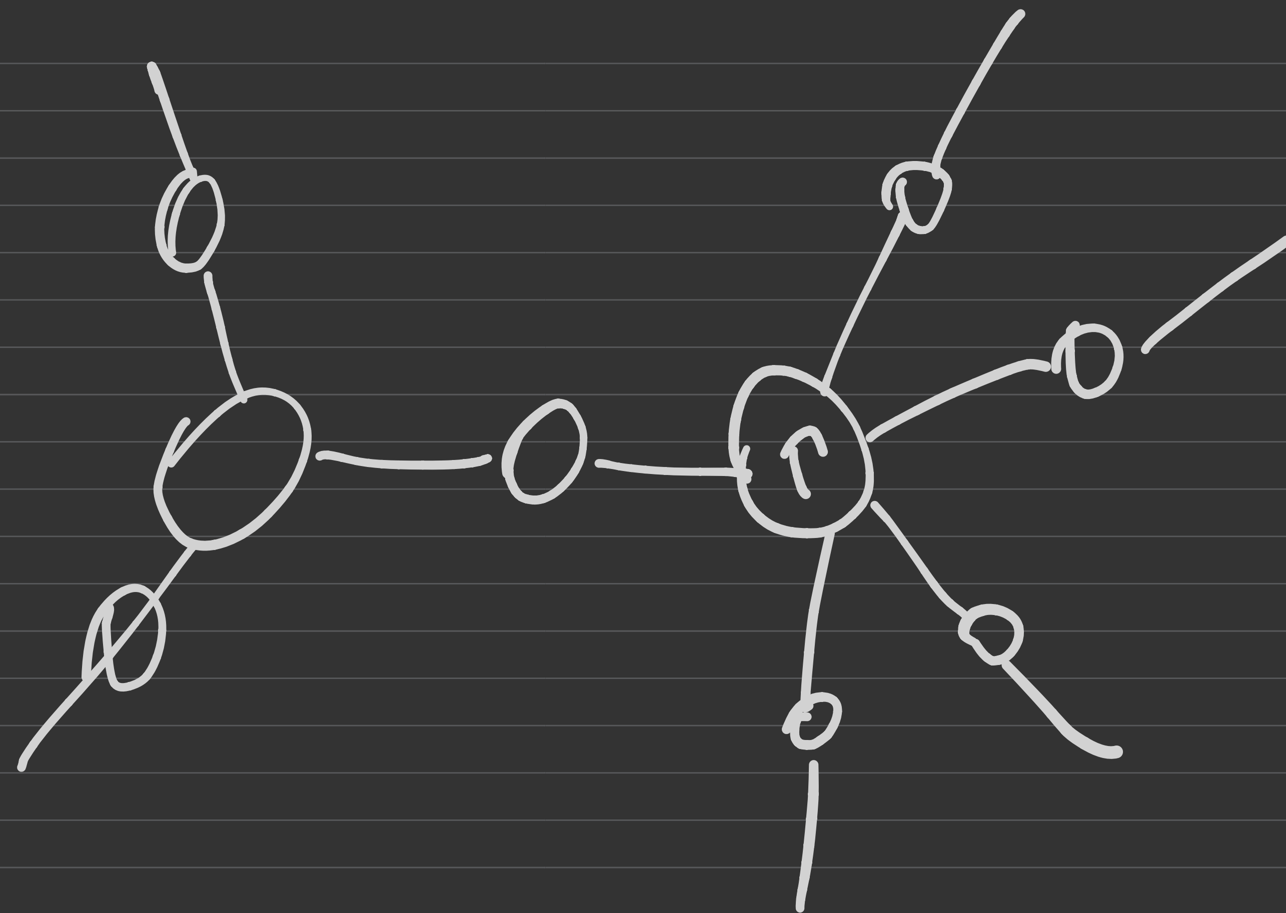
$$\underbrace{W^{(4)}} \cdot W^{(2)-1}\, W^{(2)-1}\, W^{(2)-1}\, W^{(2)-1}$$

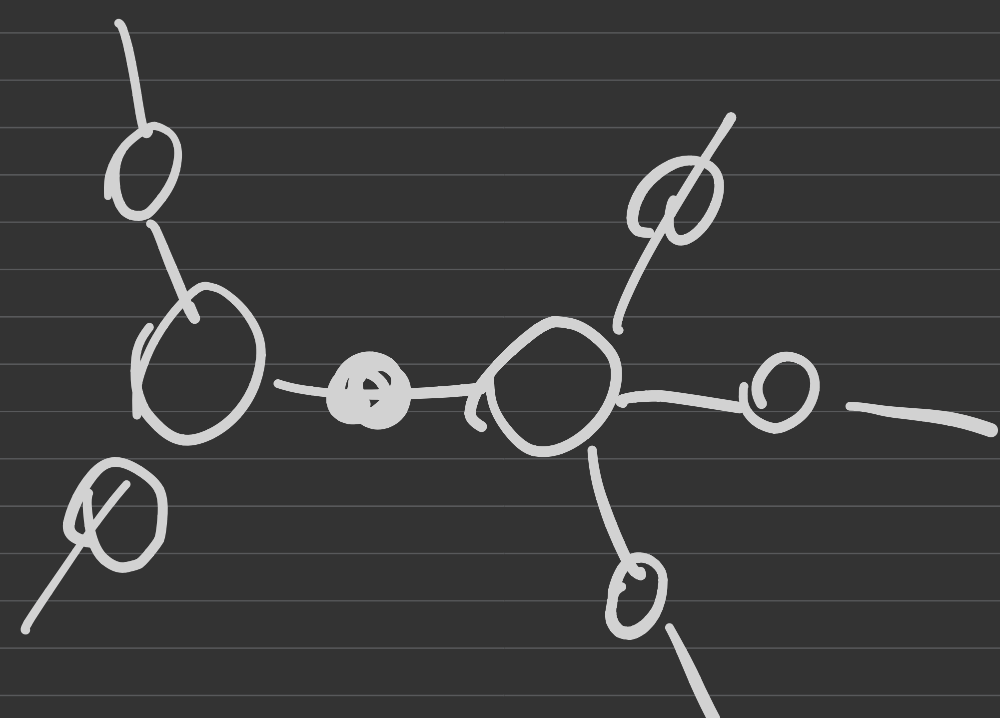
$S(2 \to 2) =$



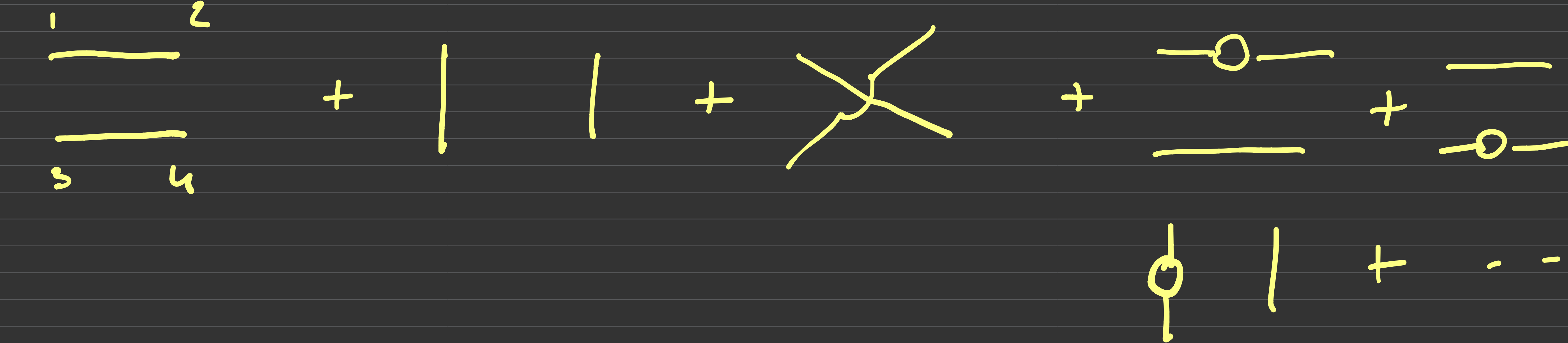
$+$ perms

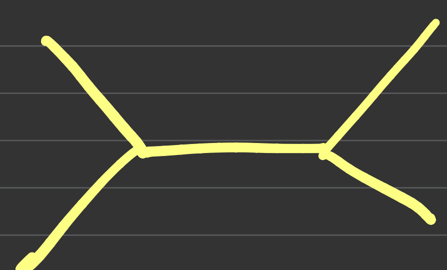
$-$





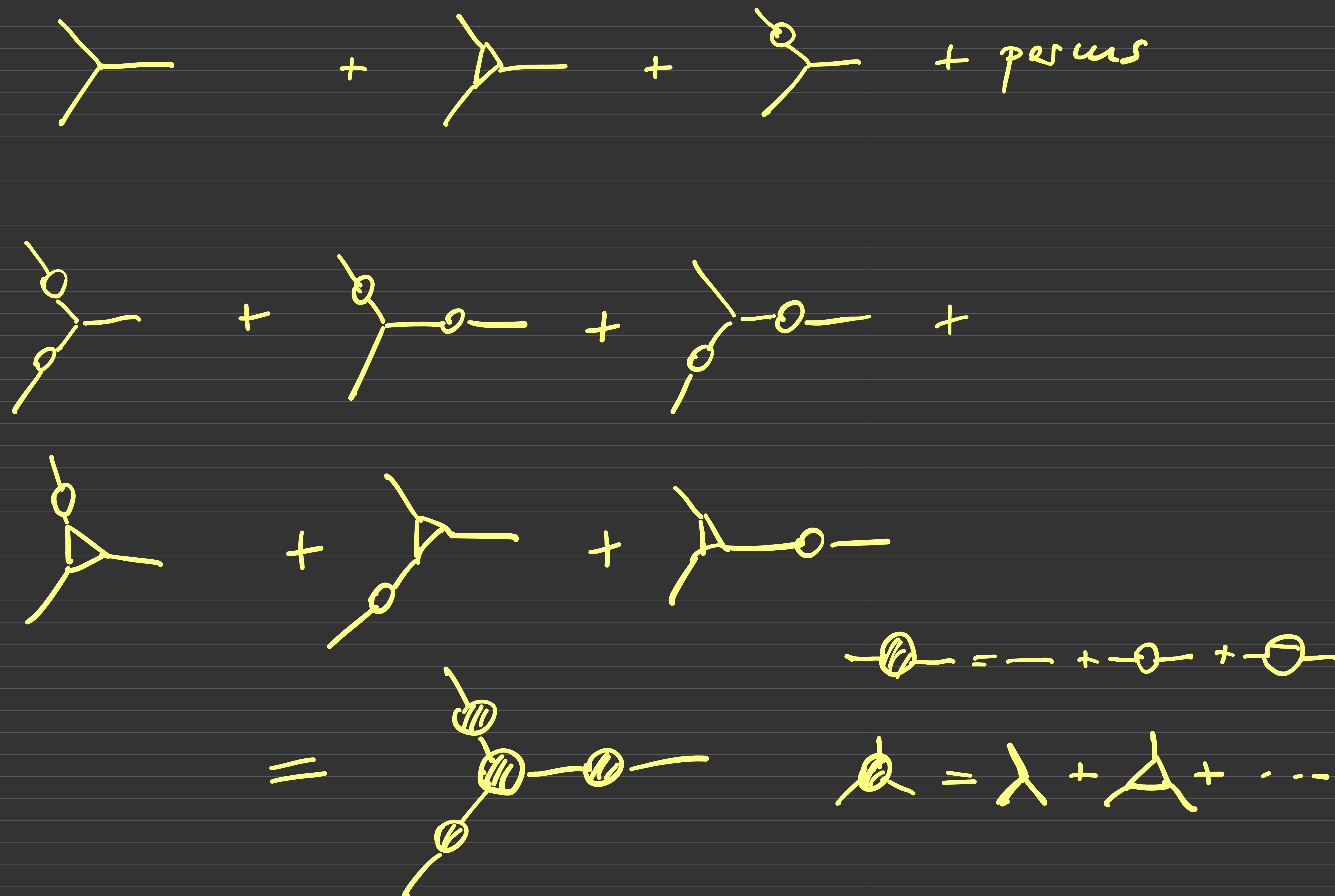
Ex $\langle \phi(x_1) \cdots \phi(x_4) \rangle$ in $\lambda \phi^3$ in pert. theory





+ perms

+ 

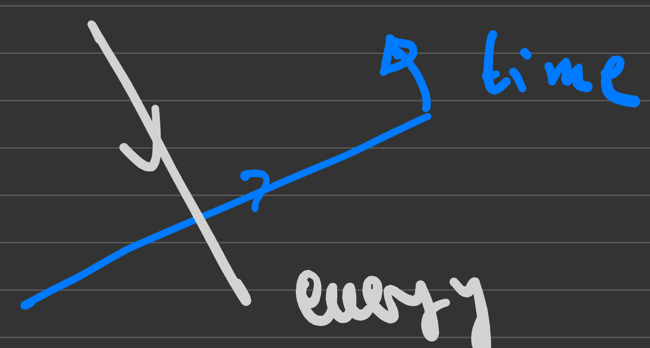

+ perms
=
λ
+ ...

△ S-matrix from Wick rotation (scalar field)

$$G_E(x_1 \dots x_n) = \int d^4p_1 \dots d^4p_n \, e^{-ip_1x_1 - \dots - ip_nx_n} \, \delta^{(4)}(p_1 + \dots p_n) \, \hat{G}_E(p_2, \dots p_n)$$

$$= \int d^4p_2 \dots d^4p_n \, e^{-ip_2(x_2-x_1) \dots - ip_n(x_n - x_1)} \, \hat{G}_E(p_2, \dots p_n)$$

$$-i(\underline{p_0 x^0} + p_1x_1 + p_2x_2 \dots)$$



$\Downarrow$ Wick rotate $\quad p_\mu x^\mu \qquad p_0 \to -ip_0$

$$\Rightarrow \int (-i)^{n-1} d^4p_2 \dots d^4p_n \, e^{-ip_2(x_2-x_1) \dots - ip_n(x_n-x_1)} \, \hat{G}_E(\hat{p}_2, \dots \hat{p}_n)$$

$$\hat{p}_\mu = (-ip_0, \vec{p})$$

- $G_M^{(n)}(p_2, \dots p_n) = (-i)^{n-1} \, G_E^{(n)}(\hat{p}_2, \dots \hat{p}_n)$

- $G_M^{(2)}(p) = -i \, G_E^{(2)}(\hat{p}) = \dfrac{-i}{(-ip_0)^2 + \vec{p}^{\,2} + m^2}$

$$= \frac{i}{p_0^2 - \vec{p}^{\,2} + m^2}$$

- LSZ $\Longrightarrow$ $\sqrt{z} = 1$

$$S(p_1 \dots p_n) = (2\pi)^4 \delta(p_1 + \dots p_n) \, G_M^{-1}(p_1) \dots G_M^{-1}(p_n) \, G_M^{(n)}(p_2, \dots p_n)$$

$$= (-i)^{-1} \cdot (2\pi)^4 \delta(p_1 + \dots p_n) \, G_E^{-1}(\hat{p}_1) \dots G_E^{-1}(\hat{p}_n) \, G_E^{(n)}(\hat{p}_2, \dots \hat{p}_n)$$

With $p_i^2 \to m^2$

- Connected part of $\hat{S}$: $\quad G^{(n)} \longrightarrow G_c^{(n)}$

$$\boxed{S_c(p_1, \dots p_n) = -i\,\Gamma_E(\hat{p}_1, \dots \hat{p}_n)}$$

Ex $\quad \lambda\phi^4 \Longrightarrow \Gamma_E^{(4)} = (2\pi)^4\,\delta^4(p_1 + \dots p_4)\,\lambda + O(\lambda^2)$

$$\Longrightarrow S'(p_1, \dots p_4) = -(2\pi)^4\,\delta^4(p_1 \dots p_4)(i\lambda) + O(\lambda^2)$$

in agreement with Feynman diagrams

for $\quad \mathcal{L} = \frac{(\partial\phi)^2}{2} - \frac{\lambda}{4!}\phi^4$

$2 \to 2$ scattering

$$S(p_1, p_2, p_3, p_4) = i\, G_E(\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4)\, G_E^{-1}(\hat{p}_1) \cdots G^{-1}(\hat{p}_4)$$

$$= -i\, \Gamma_E^{(4)}(\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4) + i\Big[\Gamma_E^{(3)}(\hat{p}_1, \hat{p}_2, \hat{q})\, G_E(\hat{q})\, \Gamma_E^{(3)}(\hat{q}, \hat{p}_3, \hat{p}_4) + $$

$$+ \text{permutations}\Big]$$

$$= -i\, \Gamma_E^{(4)} + \Big[(-i\, \Gamma_E^{(3)})(-i\, G_E(\hat{q}))(-i\, \Gamma_E^{(3)}) + \text{perms}\Big]$$

$$= i\, \Gamma_M^{(4)} + \Big[i\, \Gamma_M^{(3)}\, G_M(q)\, i\, \Gamma_M^{(3)} + \text{perms}\Big]$$

where I defined $\quad \Gamma_M^{(n)}(p_1 \cdots p_n) \equiv -\Gamma_E^{(n)}(\hat{p}_1, \ldots \hat{p}_n)$

$$\frac{\delta \Gamma}{\delta \phi_a} = 0$$ in the absence of sources

- Γ - generator of proper vertices

2-point function

$$\frac{\delta^2 \Gamma}{\delta \phi_a \delta \phi_b} = \frac{\delta}{\delta \phi_a} \frac{\delta \Gamma}{\delta \phi_b} = \frac{\delta J_b}{\delta \phi_a} = \left(\frac{\delta \phi}{\delta J} \right)^{-1}_{ba} = \left(W^{(2)^{-1}} \right)_{ab}$$

$$\phi^a = \frac{\delta W}{\delta J^a} \qquad \frac{\delta \phi^a}{\delta J^b} = \frac{\delta^2 W}{\delta J_a \delta J_b} = W^{(2)}_{ab}$$